

Technical report #7
presented to
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Surge tank system of a hydro-electric power plant

A surge tank used system used in hydro-electric power plants is shown in Figure 1 [ATR74].

From Figure 1 the mass of the fluid in the pipe connecting the reservoir to the surge tank #1 is

$$m_{12} = \rho a_{12} l_{12}, \quad \text{kg}, \quad (1)$$

where

$$\begin{aligned} a_{ij} &= \text{cross-sectional area of the conduit } i \leftrightarrow j, \quad \text{m}^2 \\ l_{ij} &= \text{length of the conduit } i \leftrightarrow j, \quad \text{m} \\ \rho &= \text{fluid mass density, } \frac{\text{kg}}{\text{m}^3}. \end{aligned}$$

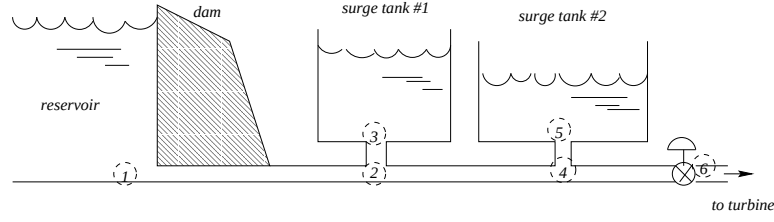


Figure 1: Surge tank system for a hydro-electric power plant

The force exerted on the mass m_{12} is

$$\frac{p_1 - p_2}{a_{12}}, \quad (2)$$

where p_i is the pressure at the point labelled i . A fluid velocity is

$$v_{12} = \frac{q_{12}}{a_{12}}, \quad \frac{\text{m}}{\text{sec}}, \quad (3)$$

where v_{ij} is the fluid velocity from point i to j .

At point labelled 3 we have

$$p_3 = \rho g h_1 = \rho g \frac{V_1}{A_1} = \frac{\rho g}{A_1} \int q_{23} dt, \quad \frac{\text{N}}{\text{m}^2} \quad (4)$$

where

$$\begin{aligned} h_i &= \text{the height of water in the tank labelled } i, \quad \text{m} \\ V_i &= \text{the liquid volume in the tank labelled } i, \quad \text{m}^3 \\ A_i &= \text{the area of the tank labelled } i, \quad \text{m}^2 \\ g &= \text{the local value of the acceleration of gravity,} \quad \frac{\text{m}}{\text{sec}^2}. \end{aligned}$$

From the law of continuity, at the junction labelled 2

$$q_{23} = q_{12} - q_{24}. \quad (5)$$

For the surge tank #2

$$p_5 = \rho g h_2 = \rho g \frac{V_2}{A_2} = \frac{\rho g}{A_2} \int q_{45} dt, \quad \frac{\text{N}}{\text{m}^2}. \quad (6)$$

The continuity condition at point 4 is

$$q_{45} = q_{24} - q_{46}. \quad (7)$$

p_1 is an independent variable and is the input of the system. For simplicity, assume that $p_6 = 0$.

The surge tank system can then be described by a set of four differential equations [ATR74]

$$\frac{q_{12}}{dt} = \frac{a_{12}}{\rho l_{12}} (p_1 - p_2) \quad (8)$$

$$\frac{p_3}{dt} = \frac{\rho g q_{23}}{A_1} \quad (9)$$

$$\frac{q_{24}}{dt} = \frac{a_{24}}{\rho l_{24}} (p_2 - p_4) \quad (10)$$

$$\frac{p_5}{dt} = \frac{\rho g q_{45}}{A_2} \quad (11)$$

The turbulent flow through a restriction connecting surge tank #1 with the main line, when the possibility of flow reversals is accounted for, is given by

$$q_{23} = \eta_1 \text{sst} (p_2 - p_3), \quad (12)$$

where

$$\text{sst}(x) = \text{sgn}(x) \sqrt{|x|}, \quad \text{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases},$$

and η_i is a semi-empirical coefficient for an orifice described by

$$q_{i-\text{out}} = \eta_i \sqrt{h_i}. \quad (13)$$

The resistor to surge tank #2 is

$$q_{45} = \eta_2 \text{sst} (p_4 - p_5). \quad (14)$$

The resistor at the wicket valve (gate valve) is

$$q_{46} = \eta_w \text{sst} (p_4 - p_5), \quad (15)$$

the valve opening is assumed to be kept constant.

Let the state variables be

$$x_1 = q_{12}, \quad x_2 = p_3, \quad x_3 = q_{24}, \quad x_4 = p_5.$$

Linearized system of the surge tanks

The linearized state equations of the surge tank system is [ATR74]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -R_{23} & -1 & R_{23} & 0 \\ 1 & 0 & -1 & 0 \\ R_{23} & 1 & -\left(R_{23} + \frac{R_{45}}{C}\right) & \frac{1}{C} \\ 0 & 0 & \frac{1}{C} & -\frac{1}{R_{46}C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} p_1 \\ 0 \\ \frac{R_{45}}{CR_{46}} p_6 \\ \frac{1}{CR_{46}} p_6 \end{bmatrix}, \quad (16)$$

where

$$\gamma_1 = \left(\frac{a_{12}}{\rho l_{12}} \right), \quad \gamma_2 = \left(\frac{\rho g}{A_1} \right), \quad \gamma_3 = \left(\frac{a_{24}}{\rho l_{24}} \right), \quad \gamma_4 = \left(\frac{\rho g}{A_2} \right), \quad C = \left(\frac{1 + R_{45}}{R_{46}} \right),$$

and R_{ij} is an equivalent linear resistances of flow between the points i and j .

Nuclear reactor

Consider a simplified nuclear reactor ([Lef65], Appendix B) where

$$\dot{\eta} = k\eta, \quad (17)$$

where η is the mean neutron density of a nuclear reactor, the reactivity k is a function of the state of the reactor and is assumed to be a linear function of η and the temperatures $y_1, \dots, y_n$ of various components of the reactor. Let

$$k = k_0 + c^T y - \rho\eta, \quad (18)$$

where k_0 is a scalar constant and $y^T = [y_1 \ \cdots \ y_n]$. Assuming that the heat transfer arises from conduction, then Newton's law of cooling gives

$$\dot{y} = Ay - b\eta, \quad (19)$$

where A is assumed to be nonsingular.

At a critical point

$$Ay - b\eta = 0, \quad \text{and} \quad (k_0 + c^T y - \rho\eta)\eta = 0. \quad (20)$$

When the reactor is shut down,

$$y_1 = 0, \quad \eta_1 = 0, \quad (21)$$

The only critical points at the steady state operating point is

$$y_2 = A^{-1}b(\rho - c^T A^{-1}b)^{-1} k_0, \quad \eta_2 = (\rho - c^T A^{-1}b)^{-1} k_0, \quad (22)$$

where $\rho - c^T A^{-1}b \neq 0$ and $\eta, \eta_2 \geq 0$.

Changing the origin to (y_2, η_2) implies the following changes in coordinates

$$\begin{aligned} x &= y - y_2 = y - A^{-1}b(\rho - c^T A^{-1}b)^{-1} k_0 \\ \theta &= \eta - \eta_2 = \eta - (\rho - c^T A^{-1}b)^{-1} k_0. \end{aligned} \quad (23)$$

Consequently the kinetic equations are

$$\dot{x} = Ax - b\theta \quad (24)$$

$$\dot{\theta} = k(\theta + \eta_2) \quad (25)$$

$$k = c^T x - \rho\theta, \quad (26)$$

where $\eta = \theta + \eta_2 > 0$

Notice also that this equation may also be written as

$$\dot{x} = Ax - b\eta_2(e^\sigma - 1) \quad (27)$$

$$\dot{\sigma} = c^T x - \rho\eta_2(e^\sigma - 1), \quad (28)$$

with a change of the a state variable

$$\sigma = \log \left(\frac{\theta + \eta_2}{\eta_2} \right) \quad (29)$$

which is in the standard form for indirect control with $\phi(\sigma) = \eta_2(e^\sigma - 1)$.

Pendulum with vibration

Consider a suspended pendulum when the suspension point is subjected to vertical vibrations of small amplitude and high frequency. [Kha96] (Example 8.10) The pendulum is shown in Figure 2. Let the motion of the suspension point be described by $a \sin \omega t$, where a is the amplitude and ω

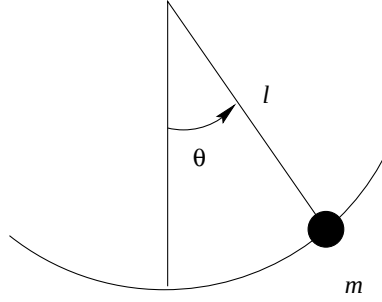


Figure 2: *Suspended pendulum*

the frequency. Then the coordinates of the bob are

$$x = l \sin \theta, \quad y = l \cos \theta - a \sin \omega t. \quad (30)$$

The velocity of the bob in the tangential direction is

$$l\dot{\theta} - a\omega \cos \omega t \sin \theta$$

and the acceleration in the tangential direction is

$$l\ddot{\theta} - a\omega^2 \sin \omega t \sin \theta.$$

Assume that

$$\frac{a}{l} \ll 1, \quad \frac{\omega_0}{\omega} \ll 1,$$

where

$$\omega_0 = \sqrt{\frac{g}{l}}$$

is the frequency of free oscillation of the pendulum in the vicinity of the lower equilibrium position $\theta = 0$. The equation of motion in a new time scale $\tau = \omega t$ is

$$\frac{d^2\theta}{d\tau^2} + \alpha\beta\epsilon\frac{d\theta}{d\tau} + (\alpha^2\epsilon^2 - \epsilon \sin \tau) \sin \theta + \alpha\beta\epsilon^2 \cos \tau \sin \theta = 0, \quad (31)$$

where

$$\epsilon = \frac{a}{l}, \quad \frac{\omega_0}{\omega} = \alpha\epsilon, \quad \alpha = \frac{\omega_0 l}{\omega a}, \quad \beta = \frac{k}{m\omega_0}.$$

The state equation is

$$\frac{dx}{d\tau} = \epsilon \begin{bmatrix} x_2 - \cos \tau \sin x_1 \\ -\alpha\beta x_2 - \alpha^2 \sin x_1 + \cos \tau x_2 \cos x_1 - \cos^2 \tau \sin x_1 \cos x_1 \end{bmatrix}, \quad (32)$$

where the state variables are

$$x_1 = \theta \tag{33}$$

$$x_2 = \frac{1}{\epsilon} \frac{d\theta}{d\tau} + \cos \tau \sin \theta \tag{34}$$

A system with unknown parameters

Instead of the system [Kha96] (Exercise 1.17(4))

$$\left. \begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1 (|x_1| + |x_2|), \\ \dot{x}_2 &= -2x_1 + x_2 - x_2 (|x_1| + |x_2|), \end{aligned} \right\} \quad (35)$$

consider an alternative system with eight unknown parameters

$$\left. \begin{aligned} \dot{x}_1 &= a_1 x_1 + a_2 x_2 - x_1 (a_3 |x_1| + a_4 |x_2|), \\ \dot{x}_2 &= a_5 x_1 + a_6 x_2 - x_2 (a_7 |x_1| + a_8 |x_2|) + u, \end{aligned} \right\}. \quad (36)$$

Indirect control system with discontinuity

Consider the indirect control system with discontinuity [Lef65] (page 79)

$$\begin{cases} \dot{y} & -ky + \varphi(\sigma) \\ \dot{\sigma} & cy - \rho\varphi(\sigma) \end{cases}, \quad (37)$$

where $k > 0$ and

$$\varphi(\sigma) = \begin{cases} M, & \sigma > 0, \\ -M, & \sigma < 0. \end{cases} \quad (38)$$

There are four cases of response depending on the value of parameters, these four case are

1. $\rho k > c > 0$
2. $c > \rho k > 0$
3. $\rho k < c < 0$
4. $c < \rho k < 0$

Indirec control with a more complicated discontinuity

Consider the same problem described by Equation 37 but here let [Lef65] (page 81)

$$\varphi(\sigma) = \begin{cases} M, & \sigma\alpha > 0, \\ 0, & -\alpha < \sigma < +\alpha, \\ -M, & \sigma < -\alpha. \end{cases} \quad (39)$$

multiple feedback system with switching lines

Consider the system described by [Lef65] (page 85)

$$\left. \begin{array}{l} \dot{x}_1 = -x_1 + \frac{3}{2}u_1 + \frac{1}{2}u_2 \\ \dot{x}_2 = -2x_2 + 2u_1 + 4u_2 \end{array} \right\}, \quad (40)$$

where $u_1, u_2 = \pm 1$

Single-link manipulator with flexible joints

5th June 1998

Consider the nonlinear dynamic equations for a single-link manipulator with flexible joints when damping is ignored

$$I\ddot{q}_1 + MgL \sin q_1 + k(q_1 - q_2) = 0 \quad (41)$$

$$J\ddot{q}_2 - k(q_1 - q_2) = u, \quad (42)$$

where I and J are moments of inertia, M is the total mass, g a gravitational constant, k a spring constant, q_1 and q_2 are angular positions and u is a torque input. Let

$$q_1 = x_1, \quad \dot{q}_1 = x_2, \quad q_2 = x_3, \quad \dot{q}_2 = x_4 \quad (43)$$

and obtain the state equations

$$\dot{x}_1 = x_2 \quad (44)$$

$$\dot{x}_2 = -\frac{MgL}{I} \sin x_1 - \frac{k}{I}x_1 + \frac{k}{I}x_3 \quad (45)$$

$$\dot{x}_3 = x_4 \quad (46)$$

$$\dot{x}_4 = \frac{k}{J}x_1 - \frac{k}{J}x_3 + \frac{u}{J}. \quad (47)$$

In other words

$$\dot{x}_1 = x_2 \quad (48)$$

$$\dot{x}_2 = a_1 \sin x_1 - a_2 x_1 + a_2 x_3 \quad (49)$$

$$\dot{x}_3 = x_4 \quad (50)$$

$$\dot{x}_4 = a_3 x_1 - a_3 x_3 + a_4 u, \quad (51)$$

where $a_1 = \frac{MgL}{I}$, $a_2 = \frac{k}{I}$, $a_3 = \frac{k}{J}$ and $a_4 = \frac{1}{J}$.

For a constant input of magnitude $a = 3$ when $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$ obtain the result as shown in Figure 3.

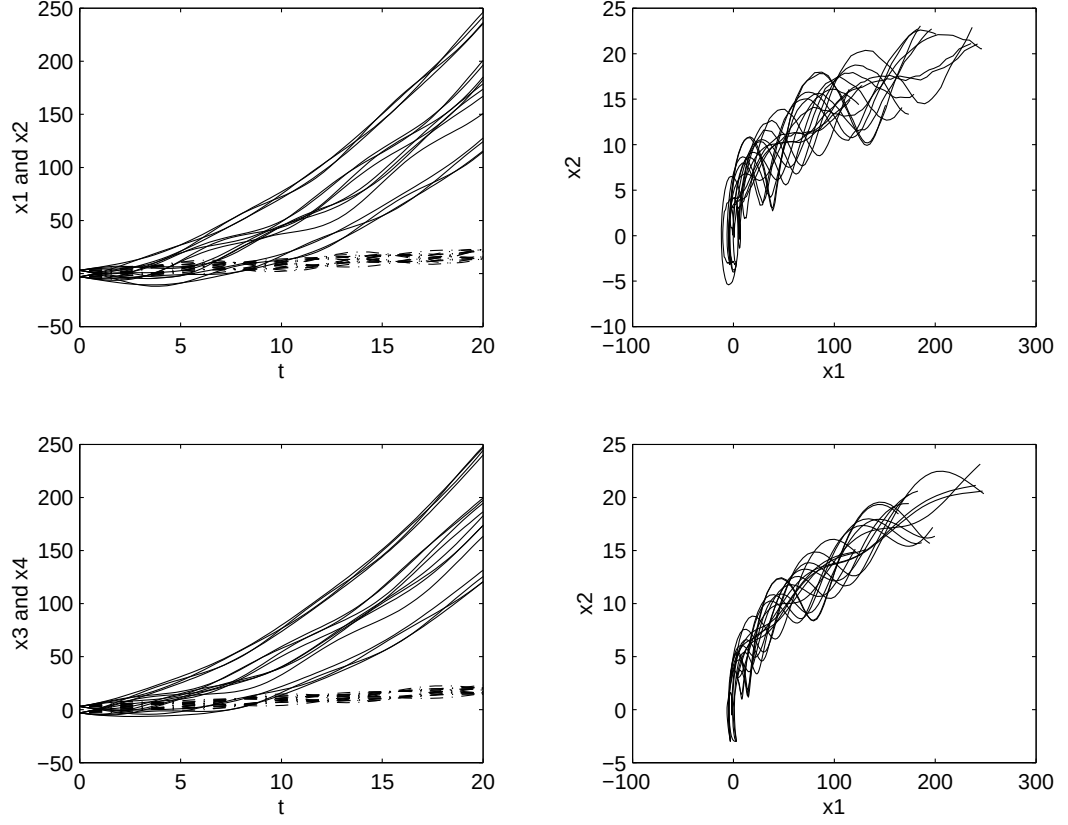


Figure 3: A single-link manipulator, $a = 3$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$, $J = 2$, $t_{CPU} = 0.3100$ sec, $t_{simulate} = 20$ sec, Initial conditions are all the $2^4 = 16$ combinations of $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$. file size = 49.97 KB (x_2 and x_4 are shown here in dash-and-dot lines.)

For a step function input of magnitude a

$$u = \begin{cases} a & t \geq 0 \\ 0 & t < 0 \end{cases}, \quad (52)$$

where $a = 3$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$. Obtain the result as shown in Figure 4.

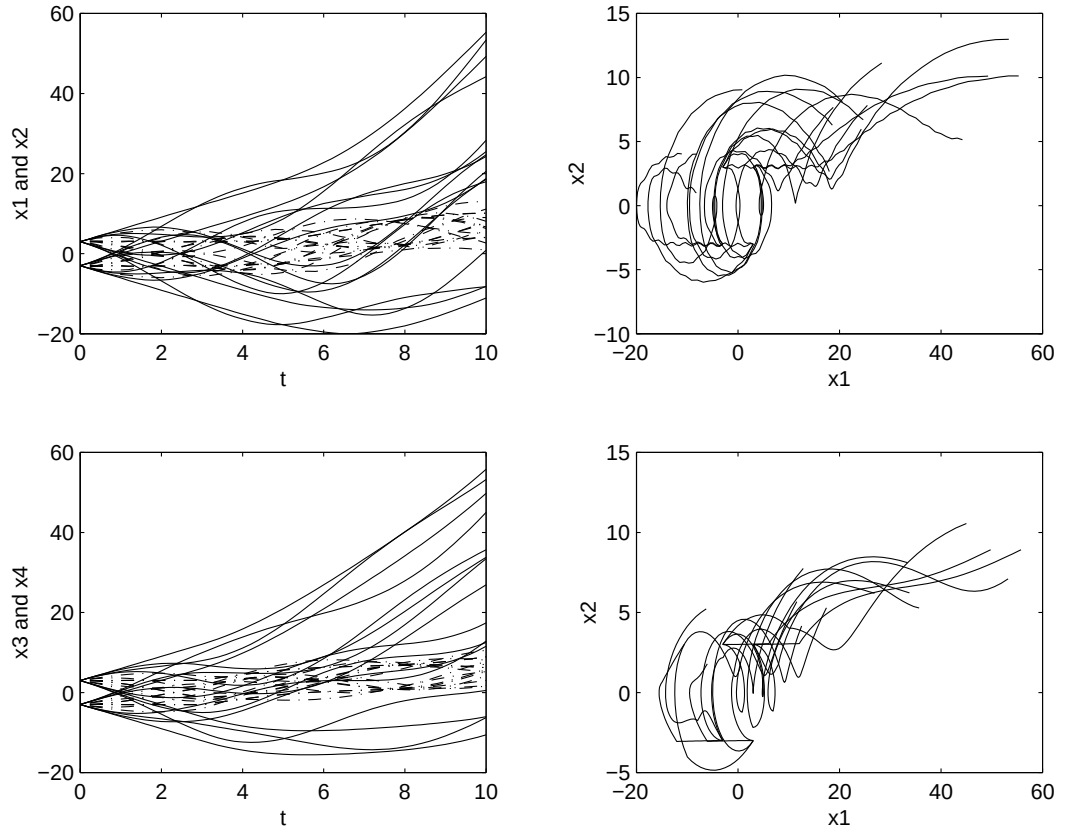


Figure 4: A single-link manipulator, $a = 3$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$, $J = 2$, $t_{CPU} = 0.2900$ sec, $t_{simulate} = 10$ sec, Initial conditions are all the $2^4 = 16$ combinations of $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$. file size = 42.29 KB (x_2 and x_4 are shown here in dash-and-dot lines.)

For a ramp function input of slope Δ

$$u = \begin{cases} \Delta t & t \geq 0 \\ 0 & t < 0 \end{cases}, \quad (53)$$

where $\Delta = 1.5$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$. Obtain the result as shown in Figure 5.

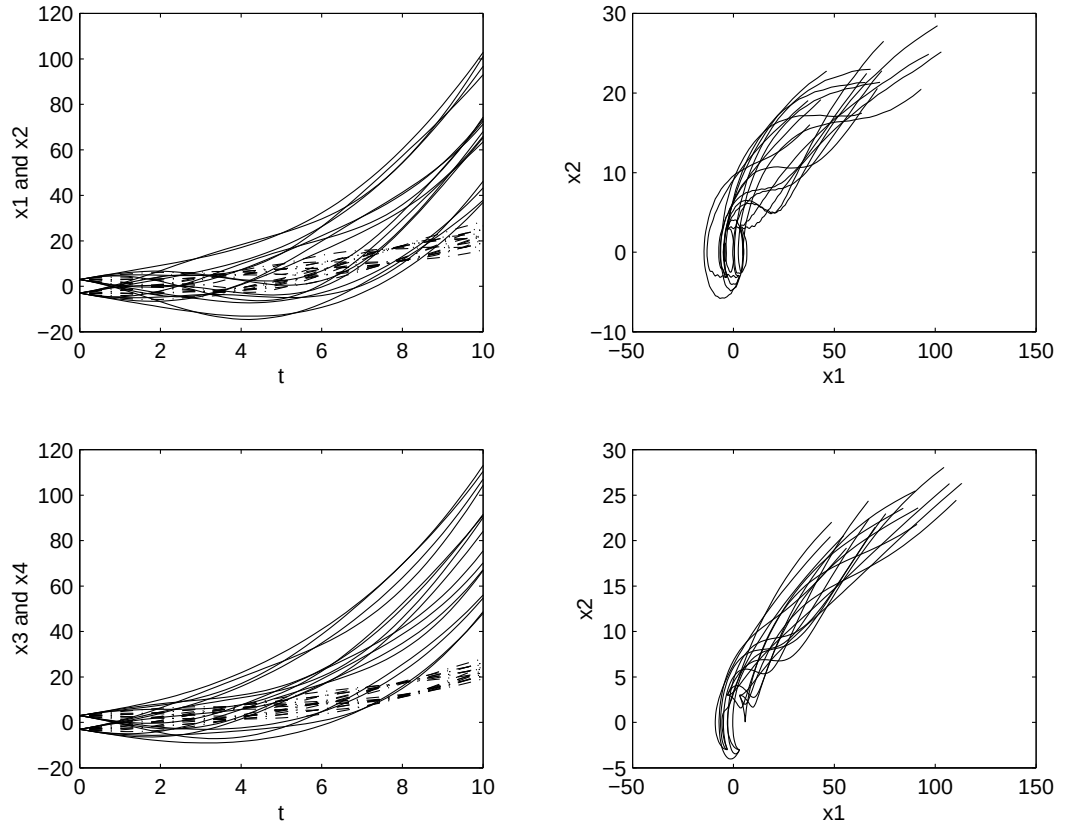


Figure 5: A single-link manipulator, $\Delta = 1.5$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$, $J = 2$, $t_{CPU} = 0.3900$ sec, $t_{simulate} = 10$ sec, Initial conditions are all the $2^4 = 16$ combinations of $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$. file size = 43.49 KB (x_2 and x_4 are shown here in dash-and-dot lines.)

For a sine function input

$$u = A \sin(\omega t) \quad (54)$$

of amplitude $A = 3$ and frequency $\omega = 2\pi$ when $\Delta = 1.5$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$ and $J = 2$, obtain the result as shown in Figure 6.

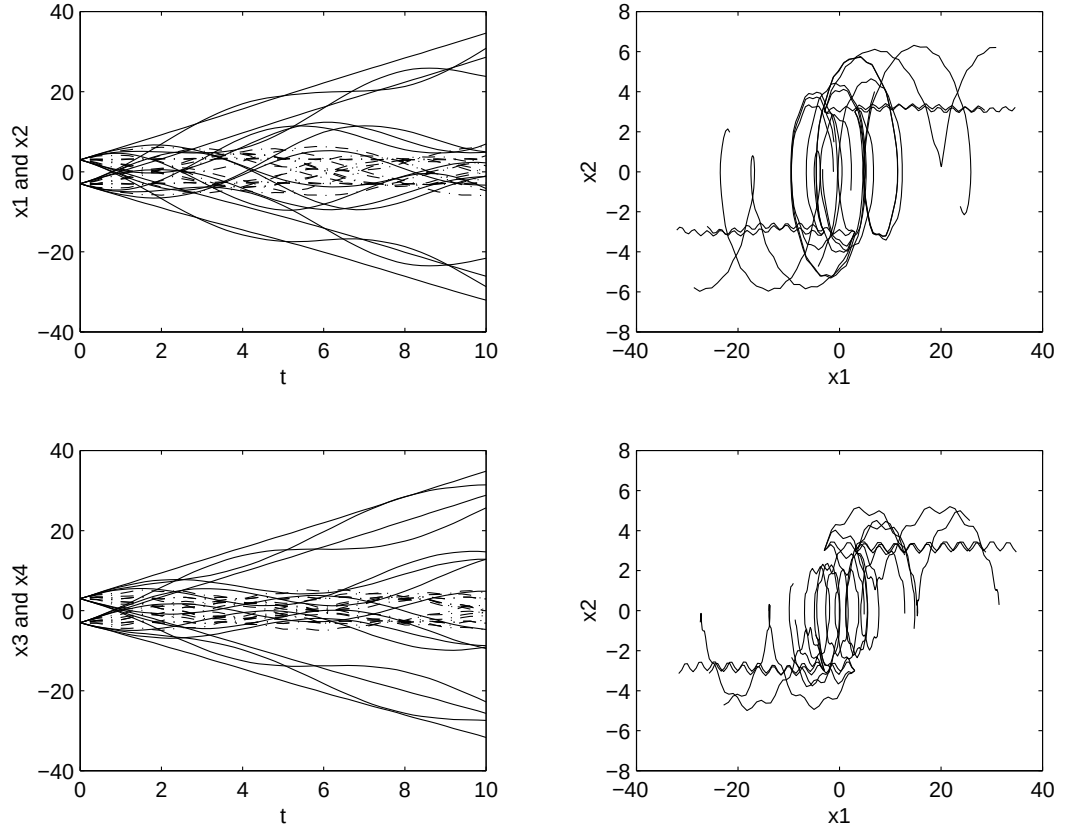


Figure 6: A single-link manipulator, $\Delta = 1.5$, $I = 1.3$, $M = 0.4$, $g = 9.8$, $L = 0.8$, $k = 0.5$, $J = 2$, $t_{CPU} = 0.2700$ sec, $t_{simulate} = 10$ sec, Initial conditions are all the $2^4 = 16$ combinations of $(x_1, x_2, x_3, x_4) = (\pm 3, \pm 3, \pm 3, \pm 3)$. file size = 42.12 KB (x_2 and x_4 are shown here in dash-and-dot lines.)

Hopfield model

Consider the Hopfield model of the artificial neural network which is based on an RC network connecting amplifiers. The characteristics of the amplifiers are

$$V_i = g_i(u_i), \quad (55)$$

where u_i and V_i are the input and output of voltages of the i^{th} amplifier. $g_i(\cdot)$ is a sigmoid function

$$g_i(\cdot) : \mathcal{R} \rightarrow (-V_M, V_M) \quad (56)$$

with asymptotes $-V_M$ and V_M . some examples of $g_i(\cdot)$ are

$$g_i(u_i) = \frac{2V_M}{\pi} \tan^{-1} \frac{\lambda \pi u_i}{2V_M}, \quad \lambda > 0 \quad (57)$$

and

$$g_i(u_i) = V_M \frac{e^{\lambda u_i} - e^{-\lambda u_i}}{e^{\lambda u_i} + e^{-\lambda u_i}}, \quad \lambda > 0, \quad (58)$$

where λ is the slope of $g_i(u_i)$ at $(u_i) = 0$.

The Kirchhoff's current law at the input node of the i^{th} amplifier gives

$$C_i \frac{du_i}{dt} = \sum_j T_{ij} V_j - \frac{1}{R_i} u_i + I_i, \quad (59)$$

where

$$\frac{1}{R_i} = \frac{1}{\rho_i} + \sum_j \frac{1}{R_{ij}}, \quad (60)$$

$C_i > 0$ and $\rho_i > 0$ are the total shunt capacitance and shunt resistance at the i^{th} amplifier input, R_{ij} is the resistor connecting the output of the j^{th} amplifier to the i^{th} input line, T_{ij} a signed conductance, $T_{ij} = \frac{1}{R_{ij}}$ or $T_{ij} = -\frac{1}{R_{ij}}$ depending on whether the output of the j^{th} amplifier is positive or negative, and I_i is a constant input current.

By letting the state variables be

$$x_i = V_i, \quad i = 1, 2, \dots, n \quad (61)$$

we have

$$\dot{x}_i = \frac{dg_i}{du_i}(u_i) \times \dot{u}_i \quad (62)$$

$$= \frac{dg_i}{du_i}(u_i) \times \frac{1}{C_i} \left(\sum_j T_{ij} x_j - \frac{1}{R_i} u_i + I_i \right). \quad (63)$$

and then

$$\dot{x}_i = \frac{1}{C_i} h_i(x_i) \left[\sum_j T_{ij} x_j - \frac{1}{R_i} g_i^{-1}(x_i) + I_i \right], \quad (64)$$

$h_i(x_i)$ being

$$h_i(x_i) \equiv \left. \frac{dg_i}{du_i}(u_i) \right|_{u_i=g_i^{-1}(x_i)} \quad (65)$$

Let $n = 2$, $V_M = 1.5$, $T_{21} = T_{12} = 1.2$, $I_i = 0$, $C_i = 0.7$, $\rho_i = 2.1$, $T_{ii} = 0$ and $g_i(u) = \left(\frac{3}{\pi}\right) \tan^{-1}\left(\frac{\lambda\pi u}{2}\right)$, $i = 1, 2$ and $\lambda = 5$.

$$h_i(x_i) = \frac{dg_i}{du}(u) \quad (66)$$

$$= \frac{6\lambda}{4 + \lambda^2\pi^2u^2} \quad (67)$$

Positioning problem

Consider the positioning problem in [Hoc91] (page 7) but here consider the case where there is an unknown parameter, namely a_1 ,

$$\dot{x}_1 = a_1 x_2 \quad (68)$$

$$\dot{x}_2 = u. \quad (69)$$

The control is bound, ie

$$-1 \leq u \leq 1, \quad \forall t. \quad (70)$$

The desired final states are

$$x_1(t_f) = x_2(t_f) = 0. \quad (71)$$

A multiplicative recurrence

Consider a multiplicative recurrence relation described as a discrete system as [Coo86] (Example 7.2)

$$x_1(t+1) = x_2(t) \quad (72)$$

$$x_2(t+1) = x_1(t)x_2(t) \quad (73)$$

Let the Lyapunov function be

$$V(x) = x_1^2 + x_2^2, \quad (74)$$

then

$$\Delta V(x(t)) = V(x(t+1)) - V(x(t)) \quad (75)$$

$$= V(f(x)) - V(x) \quad (76)$$

$$= x_2^2(t) + x_1^2(t)x_2^2(t) - x_1^2(t) - x_2^2(t) \quad (77)$$

$$= (x_2^2(t) - 1)x_1^2(t). \quad (78)$$

Therefore the domain of attraction is

$$x_1^2 + x_2^2 < 1 \quad (79)$$

and the system is not globally stable.

The simulation result shown in Figure 7 is when initial conditions are inside the domain of attraction while Figure 8 is when they are otherwise.

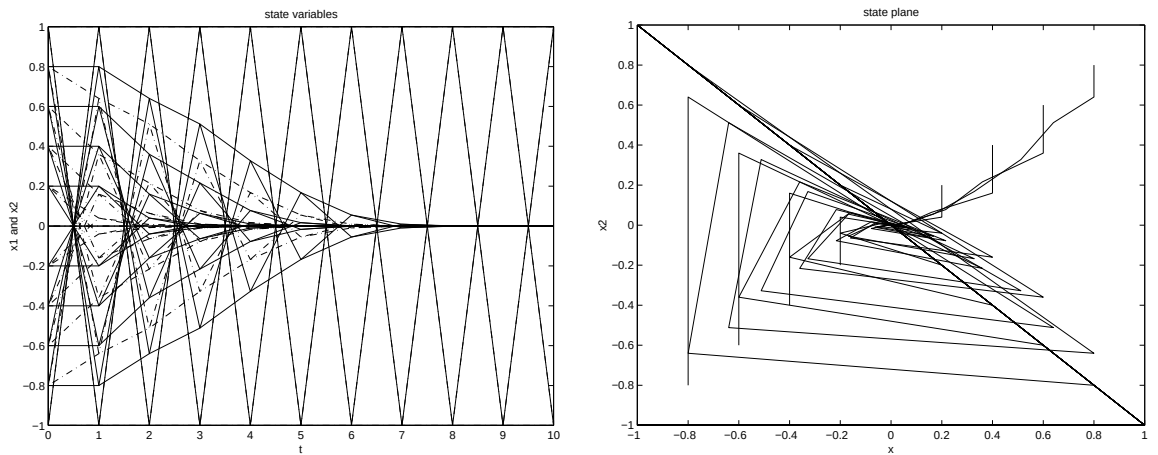


Figure 7: *Multiplicative recurrence, starting points inside the domain of attraction. Initial conditions $(x_1, x_2) = (i, i), (i, -i), \quad i = -1, -0.8, -0.6, \dots, 1$*

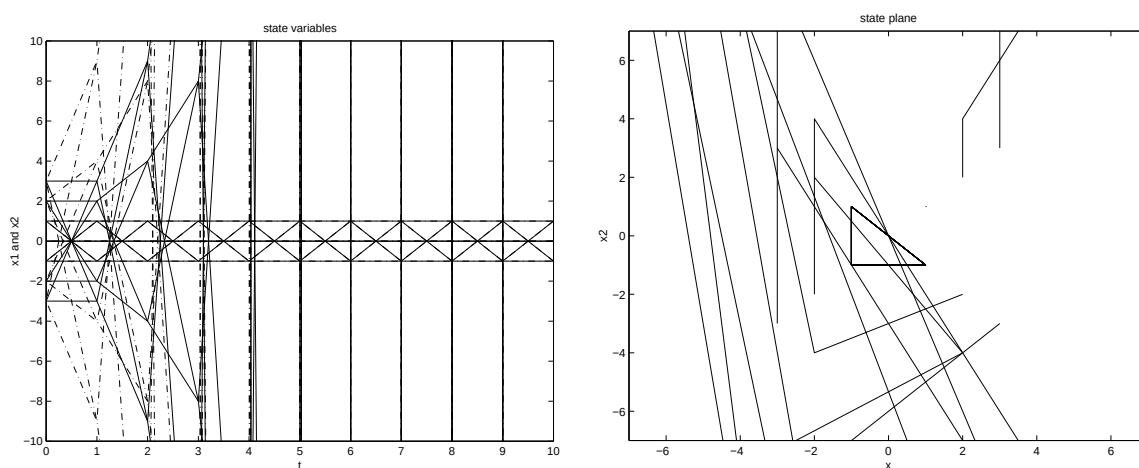


Figure 8: *Multiplicative recurrence, starting points outside the domain of attraction. Initial conditions $(x_1, x_2) = (i, i), (i, -i)$, $i = -3, -2, -1, \dots, 3$*

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